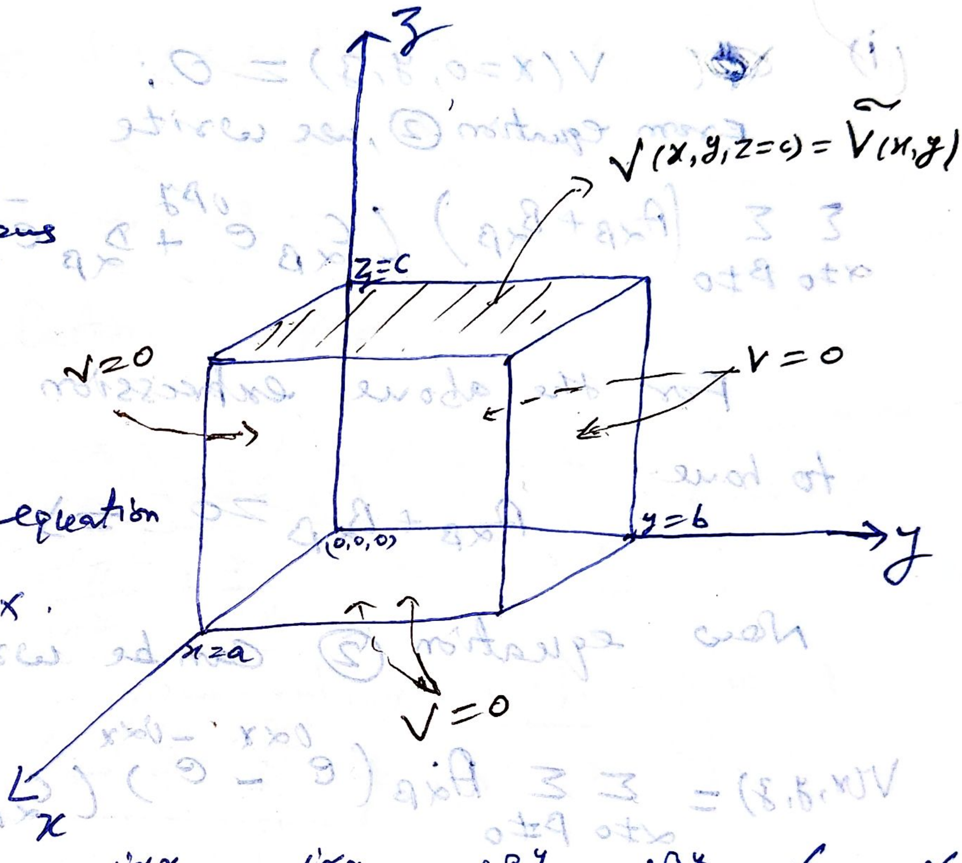
(Q) Consider a hollow retangular box with five sides at zero potential, while the sixth side $(z=c)$ has the specified potential $V(x,y,z=c)=\tilde{V}(x,y)$. (See Fig.). Find the potential inside the box. The dimensions of the box is (a,b,c) in the (x,y,z) directions.



Soln.

In the previous class note we have obtained the general solution of Laplace equation for a rectangular box.

The general is given by

$$V(x,y,z)=\sum_{\alpha\neq 0}\sum_{\beta\neq 0}\left(A_{\alpha\beta}e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x}\right)\left(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z}\right)$$

$$+\sum_{\beta\neq 0}\left(A_{0\beta}+B_{0\beta}x\right)\left(C_{0\beta}e^{i\beta y}+D_{0\beta}e^{-i\beta y}\right)\left(F_{0\beta}e^{\beta z}+G_{0\beta}e^{-\beta z}\right)$$

$$+\sum_{\alpha\neq 0}\left(A_{\alpha 0}e^{i\alpha x}+B_{\alpha 0}e^{-i\alpha x}\right)\left(C_{\alpha 0}+D_{\alpha 0}y\right)\left(F_{\alpha 0}e^{\alpha z}+G_{\alpha 0}e^{-\alpha z}\right)$$

$$+\left(A_{00}+B_{00}x\right)\left(C_{00}+D_{00}y\right)\left(F_{00}+G_{00}z\right) \quad (1)$$

where $\gamma^2=\alpha^2+\beta^2$

For the present problem with the boundary conditions that potential is zero on the five sides", the general solution can be written as

$$V(x,y,z) = \sum_{\alpha\neq 0}\sum_{\beta\neq 0}\left(A_{\alpha\beta}e^{i\alpha x} + B_{\alpha\beta}e^{-i\alpha x}\right)\left(C_{\alpha\beta}e^{i\beta y} + D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z} + G_{\alpha\beta}e^{-\gamma z}\right)$$

② of Eq. ②

Now we need to ~~e~~ evaluate the coefficients using the boundary conditions:

(i) $V(x=0, y, z) = 0$;

From equation ②, we write

$$\sum_{\alpha\neq 0}\sum_{\beta\neq 0}\left(A_{\alpha\beta}+B_{\alpha\beta}\right)\left(C_{\alpha\beta}e^{i\beta y} + D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z} + G_{\alpha\beta}e^{-\gamma z}\right) = 0$$

For the above expression to be justified, we need to have

$$A_{\alpha\beta} + B_{\alpha\beta} = 0 \Rightarrow B_{\alpha\beta} = -A_{\alpha\beta}$$

Now equation ② can be written as

$$V(x,y,z) = \sum_{\alpha\neq 0}\sum_{\beta\neq 0} A_{\alpha\beta}\left(e^{i\alpha x} - e^{-i\alpha x}\right)\left(C_{\alpha\beta}e^{i\beta y} + D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z} + G_{\alpha\beta}e^{-\gamma z}\right)$$

or $$V(x,y,z) = \sum_{\alpha\neq 0}\sum_{\beta\neq 0} A_{\alpha\beta}\sin(\alpha x)\left(C_{\alpha\beta}e^{i\beta y} + D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z} + G_{\alpha\beta}e^{-\gamma z}\right)$$

③

Again using boundary condition $V(x=a, y, z) = 0$, we obtain

~~boundary condition (ii) $V(x=a,y,z)=0$~~

From Eq. ③, ok

$$\sum_{\alpha\neq 0}\sum_{\beta\neq 0} A_{\alpha\beta}\sin(\alpha a)\left(C_{\alpha\beta}e^{i\beta y} + D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z} + G_{\alpha\beta}e^{-\gamma z}\right) = 0$$

The above expression holds true for y and z if $\alpha a = n\pi$, where $n = 0, 1, 2, 3, ---$

$$V(x,y,z) = \sum_{n\neq 0} \sum_{\beta \neq 0} A_{n\beta} \sin\left(\frac{n\pi x}{a}\right)\left(C_{n\beta} e^{\beta y} + D_{n\beta} e^{-\beta y}\right)\left(F_{n\beta} e^{\gamma z} + G_{n\beta} e^{-\gamma z}\right) \quad (4)$$

Next, using boundary conditions $V(x, y=0, z) = 0$ and $V(x, y=b, z) = 0$. [the y-direction boundary condition]

we get the similar result for y coordinate. Therefore, we write equation (4) as

$$V(x,y,z) = \sum_{n\neq 0} \sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(F_{nm} e^{\gamma z} + G_{nm} e^{-\gamma z}\right),$$

With $n, m = 0, 1, 2, 3, ---$ (5)

Now from the relation $\gamma^2 = \alpha^2 + \beta^2$

and $\alpha = \frac{n\pi}{a}$, $\beta = \frac{m\pi}{b}$, we obtain

$$\gamma = \sqrt{\frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}} = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

or $$\boxed{\gamma = \pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}} \quad (6)$$

Again the z-direction boundary condition $V(x, y, z=0) = 0$, gives.

$$V(x,y,z=0) = \sum_{n\neq 0} \sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(F_{nm} + G_{nm}\right) = 0$$

For the above expression to be true for x and y, we must have $F_{nm} + G_{nm} = 0 \Rightarrow G_{nm} = -F_{nm}$

Therefore, we can write equation ⑧

$$\tilde{V}(x,y,z) = \sum_{n\neq 0}\sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \left[e^{\gamma z} - e^{-\gamma z}\right]$$

or
$$V(x,y,z) = \sum_{n\neq 0}\sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(\gamma z)$$

From equation ⑥, we can write above equation as

$$V(x,y,z) = \sum_{n\neq 0}\sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left[\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\, z\right] \quad \text{———} ⑦$$

Again using the ~~②~~ boundary condition $V(x,y,z=c) = \tilde{V}(x,y)$, we obtain from equation ⑦

$$\tilde{V}(x,y) = \sum_{n\neq 0}\sum_{m\neq 0} A_{nm} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh\left[\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot c\right] \quad \text{———} ⑧$$

From the above equation we evaluate A_{nm} using Fourier's trick. In equation ⑧ we treat $A_{nm} \sinh\left(\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot c\right)$ as the coefficient of the Fourier series expansion of the function $\tilde{V}(x,y)$. We multiply both sides $\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$ and integrating

$$\frac{4}{ab}\int_0^a dx \int_0^b dy\, \tilde{V}(x,y) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) = A_{nm} \sinh\left[\pi\sqrt{\frac{n^2}{a^2}+\frac{m^2}{b^2}}\cdot c\right]$$

~~$\sum_{n\neq}\sum_{m\neq 0}$~~

$$A_{nm} = \frac{4}{ab\sinh\left[\pi\sqrt{n^2/a^2+m^2/b^2}\cdot c\right]}\int_0^a dx\int_0^b dy\, \tilde{V}(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

Therefore the Solution is given by

$$\boxed{V(x,y,z) = \sum_{n\neq 0}\sum_{m\neq 0}\left\{\frac{4}{ab\sinh\left[\pi\sqrt{n^2/a^2+m^2/b^2}\cdot c\right]}\int_0^a dx\int_0^b dy\, \tilde{V}(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\right\} \times \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\pi\sqrt{n^2/a^2+\frac{m^2}{b^2}}\cdot z\right)}$$

This is the Potential everywhere inside the box with the given boundary conditions.